

An algorithm to compute the differential equations for the logarithm of a polynomial

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Abstract

We present an algorithm to compute the annihilator of (i.e., the linear differential equations for) the multi-valued analytic function $f^\lambda(\log f)^m$ in the Weyl algebra D_n for a given non-constant polynomial f , a non-negative integer m , and a complex number λ . This algorithm essentially consists in the differentiation with respect to s of the annihilator of f^s in the ring $D_n[s]$ and ideal quotient computation in D_n . The obtained differential equations constitute what is called a holonomic system in D -module theory. Hence combined with the integration algorithm for D -modules, this enables us to compute a holonomic system for the integral of a function involving the logarithm of a polynomial with respect to some variables.

1 Introduction

For a given function u , it is an interesting problem both in theory and in practice to determine the differential equations which u satisfies. Let us restrict our attention to linear differential equations with polynomial coefficients. Then our problem can be formulated as follows: Let D_n be the Weyl algebra, i.e., the ring of differential operators with polynomial coefficients in the variables $x = (x_1, \dots, x_n)$. An element P of D_n is expressed as a finite sum

$$P = \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha, \beta} x^\alpha \partial^\beta, \quad (1)$$

with $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ and $a_{\alpha, \beta} \in \mathbb{C}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ are multi-indices with $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\partial_i = \partial/\partial x_i$ ($i = 1, \dots, n$) denote derivations. The *annihilator* of u (in D_n) is defined to be

$$\text{Ann}_{D_n} u = \{P \in D_n \mid Pu = 0\},$$

which is a left ideal of D_n . Since D_n is a non-commutative Noetherian ring, there exist a finite number of operators $P_1, \dots, P_N \in D_n$ which generate $\text{Ann}_{D_n} u$ as left ideal. Thus we can regard the system

$$P_1 u = \cdots = P_N u = 0$$

of linear (partial or ordinary) differential equations as a maximal one that u satisfies.

As to systems of linear differential equations, there is a notion of holonomicity, or being *holonomic*, which plays a central role in D -module theory. See Appendix for a precise definition. A holonomic system of linear differential equations admits only a finite number of linearly independent solutions although it is not a sufficient condition for holonomicity. A *holonomic function* is by definition a function which satisfies a holonomic system.

The importance of the holonomicity lies in, in addition to the finiteness property above, the fact that it is preserved under basic operations on functions such as sum, product, restriction and integration. Hence starting from some basic holonomic functions we can construct various holonomic functions by using such operations.

As one of basic holonomic functions, let us consider f^λ with a non-constant polynomial f in $x = (x_1, \dots, x_n)$ and a complex number λ . Then the function f^λ is holonomic and there is an algorithm to compute its annihilator strictly ([9],[3],[13]).

Our purpose is to give an algorithm to compute the annihilator of $f^\lambda(\log f)^m$ with a positive integer m and to prove that it is a holonomic function. This is achieved by differentiation with respect to the parameter s of the annihilator of f^s in $D_n[s]$. This method can be extended to functions of the form $f_1^{\lambda_1} \cdots f_N^{\lambda_N} (\log f_1)^{m_1} \cdots (\log f_N)^{m_N}$ for polynomials f_k , complex numbers λ_k and nonnegative integers m_k .

Since the algorithm yields a holonomic system, we can apply the integration algorithm for D -modules (see [11], [13]) to get a holonomic system for the integral of a function involving the logarithm of a polynomial.

2 Annihilators with a parameter

Let f be a non-constant polynomial in n variables $x = (x_1, \dots, x_n)$ with coefficients in the field \mathbb{C} of the complex numbers. From an algorithmic viewpoint, we assume that the coefficients of f belong to a computable field.

First, we consider formal functions of the form $f^s(\log f)^k$ with an indeterminate s . More precisely, for a non-negative integer m , we introduce the module

$$\mathcal{L}(f, m) := \bigoplus_{k=0}^m \mathbb{C}[x, f^{-1}, s] f^s(\log f)^k,$$

of which $f^s(\log f)^k$ are regarded as a free basis over $\mathbb{C}[x, f^{-1}, s]$. Then $\mathcal{L}(f, m)$ has a natural structure of left $D_n[s]$ -module, which is induced by the action of the derivation $\partial_j = \partial/\partial x_j$ defined by, for $a \in \mathbb{C}[x, f^{-1}, s]$,

$$\partial_j \{a f^s(\log f)^k\} = \left(\frac{\partial a}{\partial x_j} + s a f^{-1} \frac{\partial f}{\partial x_j} \right) f^s(\log f)^k + k a f^{-1} \frac{\partial f}{\partial x_j} f^s(\log f)^{k-1} \quad (j = 1, \dots, n)$$

if $k \geq 1$ and

$$\partial_j(a f^s) = \left(\frac{\partial a}{\partial x_j} + s a f^{-1} \frac{\partial f}{\partial x_j} \right) f^s \quad (j = 1, \dots, n).$$

In view of this action, it is easy to see that $\mathcal{L}(f, m)/\mathcal{L}(f, m-1)$ is isomorphic to $\mathcal{L}(f, 0) = \mathbb{C}[x, f^{-1}, s] f^s$ as a left $D_n[s]$ -module.

Now consider the left $D_n[s]$ -submodule

$$\mathcal{P}(f, m) := D_n[s]f^s + \cdots + D_n[s](f^s(\log f)^m)$$

of $\mathcal{L}(f, m)$. Our purpose is to determine the annihilator module

$$\text{Ann}_{D_n[s]}(f^s, \dots, f^s(\log f)^m) := \left\{ P = (P_0, P_1, \dots, P_m) \in D_n[s]^{m+1} \mid \sum_{k=0}^m P_k(f^s(\log f)^k) = 0 \right\}$$

and the annihilator ideal

$$\text{Ann}_{D_n[s]}f^s(\log f)^m := \{P \in D_n[s] \mid P(f^s(\log f)^m) = 0\}.$$

Note that there are isomorphisms

$$\begin{aligned} \mathcal{P}(f, m) &\simeq D_n[s]^{m+1} / \text{Ann}_{D_n[s]}(f^s, \dots, f^s(\log f)^m), \\ D_n[s](f^s(\log f)^m) &\simeq D_n[s] / \text{Ann}_{D_n[s]}(f^s(\log f)^m). \end{aligned}$$

Now let us regard $f^s \log f$ as a multi-valued analytic function in (x, s) on $\{(x, s) \in \mathbb{C}^{n+1} \mid f(x) \neq 0\}$.

Lemma 1 *Let $f \in \mathbb{C}[x]$ be a non-constant polynomial. Then for $a_i(x) \in \mathbb{C}[x]$,*

$$\sum_{i=0}^m a_i(x)(\log f)^i = 0$$

holds as analytic function if and only if $a_i(x, s) = 0$ for all i .

Proof: We argue by induction on m . Let $x_0 \in \mathbb{C}^n$ be a non-singular point of the hyper-surface $f(x) = 0$, i.e, assume

$$f(x_0) = 0, \quad \frac{\partial f}{\partial x_i}(x_0) \neq 0 \quad \text{for some } i \text{ with } 1 \leq i \leq n.$$

In view of the uniqueness of analytic continuation, we have only to show that each $a_i(x)$ vanishes near x_0 . Hence we may suppose that $a_i(x)$ are analytic near $x_0 = 0$ and $f(x) = x_1$. That is,

$$a_0(x) + a_1(x) \log x_1 + \cdots + a_m(x)(\log x_1)^m = 0 \tag{2}$$

holds on a neighborhood U of 0. Fix a point $x = (x_1, \dots, x_n)$ in U such that $x_1 \neq 0$. By analytic continuation along a circle $(e^{\sqrt{-1}t}x_1, x_2, \dots, x_n)$ with $0 \leq t \leq 2\pi$, the identity (2) is transformed to

$$a_0(x) + a_1(x)(\log x_1 + 2\pi\sqrt{-1}) + \cdots + a_m(x)(\log x_1 + 2\pi\sqrt{-1})^m = 0.$$

By subtraction, we get an identity of the form

$$b_0(x) + b_1(x) \log x_1 + \cdots + b_{m-1}(x)(\log x_1)^{m-1} = 0$$

with

$$b_{m-1}(x) = 2m\pi\sqrt{-1}a_m(x).$$

From the induction hypothesis it follows that $b_0(x) = \cdots = b_{m-1}(x) = 0$, which implies $a_m(x) = 0$. We are done by induction on m . \square

3 Computation of the annihilator

Now let us describe an algorithm for computing the annihilator of $f^s(\log f)^m$.

Algorithm 1 Input: a non-constant polynomial f in the variables $x = (x_1, \dots, x_n)$ with coefficients in a computable subfield of \mathbb{C} , a non-negative integer m .

- (1) Let $G = \{P_1(s), \dots, P_k(s)\}$ be a generating set of the left ideal $\text{Ann}_{D_n[s]} f^s := \{P(s) \in D_n[s] \mid P(s)f^s = 0\}$ by using an algorithm of [9] or [3] (see also [5]).
- (2) Let $e_0 = (1, 0, \dots, 0), \dots, e_m = (0, \dots, 0, 1)$ be the canonical unit vectors of \mathbb{C}^{m+1} . For each $i = 1, \dots, k$ and $j = 0, 1, \dots, m$, set

$$P_i(s)^{(j)} := \sum_{\nu=0}^j \binom{j}{\nu} \frac{\partial^{j-\nu} P_i(s)}{\partial s^{j-\nu}} e_\nu.$$

Output: $G' := \{P_i(s)^{(j)} \mid 1 \leq i \leq k, 0 \leq j \leq m\}$ generates $\text{Ann}_{D_n[s]}(f^s, \dots, f^s(\log f)^m)$.

Algorithm 2 Input: a non-constant polynomial f in the variables $x = (x_1, \dots, x_n)$ with coefficients in a computable subfield of \mathbb{C} , a non-negative integer m .

- (1) Let G' be the output of Algorithm 1.
- (2) Compute a Gröbner base G'' of the module generated by G' with respect to a term order \prec for $(D_n[s])^{m+1}$ such that $Me_j \prec M'e_k$ for any monomial M and M' if $k < j$. Let G_0 be the set of the last component of each element of G'' .

Output: G_0 generates $\text{Ann}_{D_n[s]} f^s(\log f)^m$.

Lemma 2 Let I be a left ideal of $D_n[s]$ generated by $\{P_1(s), \dots, P_k(s)\}$. For $P(s) \in D_n[s]$, and $j \in \mathbb{N}$, set

$$P(s)^{(j)} := \sum_{\nu=0}^j \binom{j}{\nu} \frac{\partial^{j-\nu} P(s)}{\partial s^{j-\nu}} e_\nu.$$

Then the left $D_n[s]$ -submodule of $(D_n[s])^{m+1}$ which is generated by $\{P(s)^{(j)} \mid P(s) \in I, 0 \leq j \leq m\}$ coincides with the one which is generated by $\{P_i(s)^{(j)} \mid 0 \leq i \leq k, 0 \leq j \leq m\}$ for any integer $m \geq 0$.

Proof: Let \mathcal{N} be the left $D_n[s]$ -module generated by $\{P_i(s)^{(j)} \mid 0 \leq i \leq k, 0 \leq j \leq m\}$ and $P(s)$ be a nonzero element of I . Then there exist

$$Q_i(s) = \sum_{l=0}^{m_i} Q_{il} s^l \quad (Q_{il} \in D_n)$$

such that $P(s) = \sum_{i=1}^k Q_i(s)P_i(s)$. Then we have

$$P(s)^{(j)} = \sum_{i=1}^k \sum_{l=0}^{m_i} Q_{il} (s^l P_i(s))^{(j)}.$$

Hence we have only to show that $(s^l P_i(s))^{(j)}$ belongs to \mathcal{N} . This can be done as follows:

$$\begin{aligned}
(s^l P_i(s))^{(j)} &= \sum_{\nu=0}^j \binom{j}{\nu} \left(\frac{\partial}{\partial s} \right)^{j-\nu} (s^l P_i(s)) e_\nu \\
&= \sum_{\nu=0}^j \binom{j}{\nu} \sum_{\mu=0}^{\min\{j-\nu, l\}} \binom{j-\nu}{\mu} (l)_\mu s^{l-\mu} \left(\frac{\partial}{\partial s} \right)^{j-\nu-\mu} P_i(s) e_\nu \\
&= \sum_{\mu=0}^{\min\{j, l\}} \binom{j}{\mu} (l)_\mu s^{l-\mu} \sum_{\nu=0}^{j-\mu} \binom{j-\mu}{\nu} \left(\frac{\partial}{\partial s} \right)^{j-\mu-\nu} P_i(s) e_\nu \\
&= \sum_{\mu=0}^{\min\{j, l\}} \binom{j}{\mu} (l)_\mu s^{l-\mu} P_i(s)^{(j-\mu)},
\end{aligned}$$

where $(l)_\mu := l(l-1)\cdots(l-\mu+1)$. \square

Theorem 1 *The output of Algorithm 1 coincides with $\text{Ann}_{D_n[s]}(f^s, \dots, f^s(\log f)^m)$.*

Proof: Let $P(s)$ belong to $\text{Ann}_{D_n[s]} f^s$. Differentiating the equation $P(s)f^s = 0$ with respect to s , we get

$$\sum_{\nu=0}^j \binom{j}{\nu} \frac{\partial^{j-\nu} P_i(s)}{\partial s^{j-\nu}} (f^s(\log f)^\nu) = 0$$

for $0 \leq j \leq m$. This shows that each $P_i(s)^{(j)}$ annihilates $(f^s, \dots, f^s(\log f)^m)$.

Set $\mathcal{M} := \text{Ann}_{D_n[s]}(f^s, \dots, f^s(\log f)^m)$. Let \mathcal{N} be the left $D_n[s]$ -module generated by the output G' of Algorithm 1. The argument above shows that \mathcal{N} is a left D_n -submodule of \mathcal{M} . Hence we have only to prove $\mathcal{N} = \mathcal{M}$. For this purpose let \mathcal{N}_j be the left $D_n[s]$ -module generated by $\{P_i(s)^{(\nu)} \mid 1 \leq i \leq k, 0 \leq \nu \leq j\}$ and set

$$\mathcal{M}_j := \{(Q_0, Q_1, \dots, Q_m) \in \mathcal{M} \mid Q_\nu = 0 \text{ if } \nu > j\}.$$

Let $Q(s) = (Q_0(s), \dots, Q_j(s), 0, \dots, 0)$ be an element of \mathcal{M}_j . Then

$$\sum_{\nu=0}^j Q_\nu(s) (f^s(\log f)^\nu) = 0$$

holds. In view of the action of $D_n[s]$ on $\mathcal{L}(f, j)$ noted in Section 1, this implies $Q_j(s)f^s = 0$. Hence $Q_j(s)^{(j)}$ belongs to \mathcal{N}_j by Lemma 2. It is easy to see that $Q(s) - Q_j(s)^{(j)}$ belongs to \mathcal{M}_{j-1} . This means $\mathcal{M}_j = \mathcal{N}_j + \mathcal{M}_{j-1}$ for $1 \leq j \leq m$. Then we can show that $\mathcal{N}_j = \mathcal{M}_j$ holds for $1 \leq j \leq m$ by induction on m noting $\mathcal{N}_0 = \mathcal{M}_0$. \square

Remark 1 If f is weighted homogeneous, i.e., if there exist rational numbers w_i such that $\sum_{i=1}^n w_i x_i \partial_i(f) = f$, then $D_n[s]f^s(\log f)^m$ is isomorphic to $\mathcal{L}(f, m)$ as left $D_n[s]$ -module. That is, we have an isomorphism

$$D_n[s]/\text{Ann}_{D_n[s]} f^s(\log f)^m \simeq (D_n[s])^{m+1}/\text{Ann}_{D_n[s]}(f^s, \dots, f^s(\log f)^m).$$

of left $D_n[s]$ -module In fact, this follows from the relations

$$\left(\sum_{i=1}^n w_i x_i \partial_i - s \right) (f^s(\log f)^k) = k f^s(\log f)^{k-1} \quad (k \geq 1).$$

4 Specialization of the parameter

Let us fix a complex number λ . (From an algorithmic view point, we assume λ lies in a computable subfield of the field \mathbb{C} .) We set

$$\mathcal{L}(f, m, \lambda) := \bigoplus_{k=0}^m \mathbb{C}[x, f^{-1}] f^\lambda (\log f)^k,$$

where $f^\lambda (\log f)^k$ are regarded as a free basis over $\mathbb{C}[x, f^{-1}]$. Substituting λ for s gives $\mathcal{L}(f, m, \lambda)$ a natural structure of left D_n -module. In fact, one has

$$\partial_j \{a f^\lambda (\log f)^k\} \left(\frac{\partial a}{\partial x_j} + \lambda a f^{-1} \frac{\partial f}{\partial x_j} \right) f^\lambda (\log f)^k + k a f^{-1} \frac{\partial f}{\partial x_j} f^\lambda (\log f)^{k-1} \quad (j = 1, \dots, n)$$

for $k \geq 1$ and

$$\partial_j (a f^\lambda) = \left(\frac{\partial a}{\partial x_j} + \lambda a f^{-1} \frac{\partial f}{\partial x_j} \right) f^\lambda \quad (j = 1, \dots, n)$$

with $a \in \mathbb{C}[x, f^{-1}]$. This implies that $\mathcal{L}(f, m, \lambda) / \mathcal{L}(f, m-1, \lambda)$ is isomorphic to $\mathcal{L}(f, 0, \lambda) = D_n f^\lambda$ as a left D_n -module. It follows that $\mathcal{L}(f, m, \lambda)$ is holonomic since so is $D_n f^\lambda$ as was proved by Bernstein [1].

Set

$$\mathcal{P}(f, m, \lambda) := D_n f^\lambda + \dots + D_n (f^\lambda (\log f)^m)$$

We define the annihilators of $(f^s, \dots, f^\lambda (\log f)^m)$ and of $f^\lambda (\log f)^m$ to be

$$\text{Ann}_{D_n}(f^\lambda, \dots, f^\lambda (\log f)^k) := \{P = (P_0, P_1, \dots, P_m) \in (D_n)^{m+1} \mid \sum_{k=0}^m P_k (f^\lambda (\log f)^k) = 0\},$$

$$\text{Ann}_{D_n} f^\lambda (\log f)^m := \{P \in D_n \mid P(f^\lambda (\log f)^m) = 0\}$$

respectively. Then we have isomorphisms

$$\begin{aligned} \mathcal{P}(f, m, \lambda) &\simeq (D_n)^{m+1} / \text{Ann}_{D_n}(f^\lambda, \dots, f^\lambda (\log f)^k) \\ D_n(f^\lambda (\log f)^m) &\simeq D_n / \text{Ann}_{D_n}(f^\lambda (\log f)^m). \end{aligned}$$

In the sequel, we need information on the integral roots of the *Bernstein-Sato polynomial* or the *b-function* of f , which is, by definition, the monic polynomial $b_f(s)$ of the least degree such that a formal functional equation

$$P(s) f^{s+1} = b_f(s) f^s \tag{3}$$

holds with some $P(s) \in D_n[s]$. The existence of such a functional equations was proved by Bernstein [1]. It was proved by Kashiwara [4] that the roots of $b_f(s) = 0$ are negative rational numbers. An algorithm to compute $b_f(s)$ and an associated operator $P(s)$ was given in [8]. The following proposition generalizes a result of Kashiwara [4, Proposition 6.2]:

Theorem 2 Let $b_f(s)$ be the Bernstein-Sato polynomial of f , i.e., a polynomial in s of the least degree such that $P(s)f^{s+1} = b_f(s)f^s$ holds with a $P(s) \in D_n[s]$. Let λ be a complex number such that $b_f(\lambda - \nu) \neq 0$ for any positive integer ν . Then we have

$$\begin{aligned}\text{Ann}_{D_n}(f^\lambda, \dots, f^\lambda(\log f)^k) &= \{P(\lambda) \mid P(s) \in \text{Ann}_{D_n[s]}(f^s, \dots, f^s(\log f)^k), \\ \text{Ann}_{D_n}f^\lambda(\log f)^m &= \{P(\lambda) \mid P(s) \in \text{Ann}_{D_n[s]}f^s(\log f)^m\}.\end{aligned}$$

Proof: We have only to show the first equality. Assume that $\sum_{k=0}^m P_k(f^\lambda(\log f)^k) = 0$ holds with $P_k \in D_n$. Then there exist non-negative integer $l \geq 0$ and polynomials $a_k(x, s) \in \mathbb{C}[x, s]$ such that

$$\sum_{k=0}^m P_k(f^s(\log f)^k) = (s - \lambda) \sum_{k=0}^m a_k(x, s) f^{s-l}(\log f)^k.$$

By using the functional equation (3), we can find an operator $Q(s) \in D_n[s]$ such that

$$b_f(s-1) \cdots b_f(s-l) f^{s-l} = Q(s) f^s.$$

In view of the action of $D_n[s]$ on $\mathcal{L}(f, m)$, there exist $a'_k(x, s) \in \mathbb{C}[x, s]$ and a non-negative integers l_1 such that

$$b_f(s-1) \cdots b_f(s-l) f^{s-l}(\log f)^m = Q(s) \{f^s(\log f)^m\} + \sum_{k=0}^{m-1} a'_k(x, s) f^{s-l_1}(\log f)^k.$$

Proceeding inductively, we conclude that there exist a polynomial $b(s) \in \mathbb{C}[s]$ which is a product (possibly with multiplicities) of $b_f(s-j)$ with $j \geq 1$ and operators $\tilde{Q}_k(s) \in D_n[s]$ such that

$$b(s) \sum_{k=0}^m a_k(x, s) f^{s-l}(\log f)^k = \sum_{k=0}^m \tilde{Q}_k(s) \{f^s(\log f)^k\}.$$

Hence

$$\tilde{P}(s) := b(s) \sum_{k=0}^m P_k e_k - (s - \lambda) \sum_{k=0}^m \tilde{Q}_k(s) e_k$$

belongs to $\text{Ann}_{D_n[s]}(f^s, \dots, f^s(\log f)^k)$ and $b(\lambda) \sum_{k=0}^m P_k e_k = \tilde{P}(\lambda)$. This completes the proof since $b(\lambda) \neq 0$ by the assumption. \square

If $b_f(\lambda - \nu) = 0$ for some positive integer ν , then set $\nu_0 := \max\{\nu \in \mathbb{Z} \mid b_f(\lambda - \nu) = 0\}$ and $\lambda_0 := \lambda - \nu_0$. Then λ_0 satisfies the condition of Theorem 2. Then for $(P_0, \dots, P_m) \in (D_n)^{m+1}$, we have

$$\begin{aligned}(P_0, \dots, P_m) &\in \text{Ann}_{D_n}(f^\lambda, \dots, f^\lambda(\log f)^k) \\ \Leftrightarrow (P_0 f^{\nu_0}, \dots, P_m f^{\nu_0}) &\in \text{Ann}_{D_n}(f^{\lambda_0}, \dots, f^{\lambda_0}(\log f)^k) \\ \Leftrightarrow (P_0, \dots, P_m) &\in \text{Ann}_{D_n}(f^{\lambda_0}, \dots, f^{\lambda_0}(\log f)^k) : f^{\nu_0}.\end{aligned}$$

The module quotient in the last line can be obtained by computing the module intersection or else by syzygy computation. Now let us describe two algorithms for module quotient in

general. First, let us define the componentwise product of two elements $P = (P_0, \dots, P_m)$ and $Q = (Q_0, \dots, Q_m)$ of $(D_n)^{m+1}$ to be $PQ := (P_0Q_0, \dots, P_mQ_m)$. Let N be a left D_n -submodule of $(D_n)^{m+1}$ and P be a nonzero element of $(D_n)^{m+1}$. Then the module quotient $N : P$ is defined to be

$$N : P := \{Q \in (D_n)^{m+1} \mid QP \in N\},$$

which is a left D_n -submodule of $(D_n)^{m+1}$.

Algorithm 3 Input: A set G_1 of generators of a left D_n -submodule N of $(D_n)^{m+1}$ and a non-zero element $P = (P_0, P_1, \dots, P_m)$ of $(D_n)^{m+1}$.

- (1) Introducing a new variable t , compute a Gröbner base G_2 of the left $D_n[t]$ -module of $(D_n[t])^{m+1}$ which is generated by $\{(1-t)P_ke_k \mid 0 \leq k \leq m\} \cup \{tQ \mid Q \in G_1\}$ with respect to a term order \prec such that $x^\alpha \partial^\beta e_j \prec te_k$ for any $j, k \in \{0, 1, \dots, m\}$ and $\alpha, \beta \in \mathbb{N}^n$.
- (2) $G_3 := G_2 \cap (D_n)^{m+1}$.
- (3) $G_4 := \{Q/P \mid Q \in G_3\}$, where Q/P denotes the element in $(D_n)^{m+1}$ such that $(Q/P)P = Q$ in the sense of componentwise product.

Output: G_4 generates the module quotient $N : P$.

In fact, we can show in the same way as in the commutative case that G_3 generates the left module $N \cap (D_n)^{m+1}P$. In particular, for each $Q \in G_3$, there exists $Q' \in (D_n)^{m+1}$ such that $Q = Q'P$. Let us denote this Q' by Q/P . Then Q' belongs to the quotient module $N : P$. Conversely, if Q' belongs to $N : P$, then $Q'P$ belongs to $N \cap (D_n)^{m+1}P$. Hence Q' belongs to the module generated by G_4 . The correctness of the following algorithm should be clear:

Algorithm 4 Input: A set $G_1 = \{Q_1, \dots, Q_k\}$ of generators of a left D_n -submodule N of $(D_n)^{m+1}$ and a non-zero element $P = (P_0, \dots, P_m)$ of $(D_n)^{m+1}$.

- (1) Compute a set G_2 of generators of the syzygy module

$$\mathcal{S} := \{(S_0, S_1, \dots, S_m, S_{m+1}, \dots, S_{m+k}) \in (D_n)^{m+k+1} \mid \sum_{j=0}^m S_j P_j e_j + \sum_{j=m+1}^{m+k} S_{m+j} Q_j = 0\}$$

via a Groebner base of the module generated by $P_j e_j$ ($0 \leq j \leq m$) and Q_j ($1 \leq j \leq k$).

- (2) Let G_3 be the set of the first $m+1$ components of the elements of G_2 .

Output: G_3 generates the module quotient $N : P$.

Summed up, the annihilators for $(f^\lambda((\log f)^k)_{0 \leq k \leq m})$ and $(f^\lambda(\log f)^m)$ are computed as follows:

Algorithm 5 Input: a non-constant polynomial f in the variables $x = (x_1, \dots, x_n)$ with coefficients in a computable subfield of \mathbb{C} , a number λ which belongs to a computable subfield of \mathbb{C} , a non-negative integer m .

- (1) Compute a set G_1 of generators of $\text{Ann}_{D_n[s]}(f^s, \dots, f^s(\log f)^k)$ by Algorithm 1.
- (2) Compute the (global) Bernstein-Sato polynomial $b_f(s)$ of f by using one of the algorithms in [8], [9], [3] or their modifications.
- (3) Let ν_0 be the largest positive integer ν such that $b_f(\lambda - \nu) = 0$ if there are any such ν . If there are no positive integer ν such that $b_f(\lambda - \nu) = 0$, then set $\nu_0 = 0$.
- (4) Set $\lambda_0 := \lambda - \nu_0$ and $G_2 := G_1|_{s=\lambda_0}$ (substitute λ_0 for s in each element of G_1).
- (5) If $\nu_0 > 0$, then let G_3 be a set of generators of the module quotient $\langle G_2 \rangle : f^{\nu_0} = \langle G_2 \rangle : (f^{\nu_0}, \dots, f^{\nu_0})$, where $\langle G_2 \rangle$ denotes the left module generated by G_2 .
- (6) If $\nu_0 = 0$, then set $G_3 := G_2$.
- (7) Compute a Gröbner base G_4 of the module generated by G_3 with respect to a term order \prec for $(D_n)^{m+1}$ such that $Me_j \prec M'e_k$ for any monomial M and M' if $k < j$. Let G_5 be the set of the last component of each element of G_4 .

Output: G_3 generates $\text{Ann}_{D_n}(f^\lambda, \dots, f^\lambda(\log f)^m)$; G_5 generates $\text{Ann}_{D_n} f^\lambda(\log f)^m$.

Remark 2 In step (3) of the algorithm above, we need only integer roots of the b -function. Hence one can employ a method described in [6] to determine all the integer roots of the b -function efficiently without computing the whole b -function.

5 Implementation and examples

We have implemented the algorithms in a computer algebra system Risa/Asir [7], which is capable of Groebner base computation of modules over the ring of differential operators as well as over the ring of polynomials.

Example 1 (one dimensional case) Let f be a square-free polynomial in one variable x with complex coefficients. Since $\text{Ann}_{D_1[s]} f^s$ is generated by $f\partial - sf'$, the annihilator module $\text{Ann}_{D_1[s]}(f^s, \dots, f^s(\log f)^m)$ is generated by $m+1$ elements

$$(f\partial_x - sf', 0, \dots, 0), \quad (-f', f\partial_x - sf', 0, \dots, 0), \quad \dots, \quad (0, \dots, 0, -mf', f\partial_x - sf')$$

with $\partial_x = d/dx$ and $f' = \partial(f)$.

Since the Bernstein-Sato polynomial of f is $s+1$, the substitution $s = \lambda$ gives generators of $\text{Ann}_{D_1}(f^\lambda, \dots, f^\lambda(\log f)^m)$ if $\lambda \neq 0, 1, 2, \dots$. In particular, $\text{Ann}_{D_1}(f^{-1}, \dots, f^{-1}(\log f)^m)$ is generated by

$$(\partial_x f, 0, \dots, 0), \quad (-f', \partial_x f, 0, \dots, 0), \quad \dots, \quad (0, \dots, 0, -mf', \partial_x f).$$

In view of Algorithm 5, we can verify that $\text{Ann}_{D_1}(1, \dots, (\log f)^m) = \text{Ann}_{D_1}(f^{-1}, \dots, f^{-1}(\log f)^m) :$ f is generated by

$$(\partial_x, 0, \dots, 0), \quad (-f', f\partial_x, 0, \dots, 0), \quad \dots, \quad (0, \dots, 0, -mf', f\partial_x).$$

Explicit generators of $\text{Ann}_{D_1}(\log f)^m$ for $m \geq 1$ would be complicated: For example, if $f = x^3 - x$ and $m = 1$, Algorithm 5 gives generators

$$\begin{aligned} & (3x^5 - 4x^3 + x)\partial_x^2 + (3x^4 + 1)\partial_x, \\ & (x^3 - x)\partial_x^3 + (-3x^4 + 9x^2 - 2)\partial_x^2 + (-3x^3 + 3x)\partial_x \end{aligned}$$

of $\text{Ann}_{D_1} \log f$, which is not generated by a single element.

Example 2 Set $f = x^2y^2 + z^2$ with $n = 2$ and $(x_1, x_2, x_3) = (x, y, z)$, $\partial_x = \partial/\partial x$ and so on. First $\text{Ann}_{D_3[s]}f^s$ is generated by

$$\begin{aligned} & -x\partial_x + y\partial_y, \quad y\partial_y + z\partial_z - 2s, \\ & z\partial_x - y^2x\partial_z, \quad z\partial_y - yx^2\partial_z, \\ & -z\partial_x^2 + y^3\partial_z\partial_y + y^2\partial_z. \end{aligned}$$

Since the Bernstein-Sato polynomial of f is $b_f(s) = (s+1)^3(2s+3)$, the substitution $s = -1$ gives a set of generators of $\text{Ann}_{D_3}f^{-1}\log f$. Then by ideal quotient computation we get a set of generators

$$\begin{aligned} & -x\partial_x + y\partial_y, \quad -z\partial_x + y^2x\partial_z, \\ & \partial_y^2 + x^2\partial_z^2, \quad \partial_x^2 + y^2\partial_z^2, \\ & -z\partial_y + yx^2\partial_z, \quad \partial_y\partial_x^2 - zy\partial_z^3, \\ & -\partial_y^2\partial_x + zx\partial_z^3, \quad y\partial_y\partial_x + z\partial_z\partial_x \\ & y\partial_z\partial_y + z\partial_z^2 + \partial_z, \\ & y\partial_y^2 + z\partial_z\partial_y + \partial_y, \\ & z\partial_y\partial_x + zyx\partial_z^2 - yx\partial_z, \\ & \partial_y^2\partial_x^2 + z^2\partial_z^4 + 2z\partial_z^3 \end{aligned}$$

of $\text{Ann}_{D_3} \log f$. Let us consider the integral

$$u(t) := \int_{\mathbb{R}^3} e^{-t(x^2+y^2+z^2)} \log(x^2y^2 + z^2) dx dy dz,$$

which is well-defined for $t > 0$. Then $u(t)$ satisfies ordinary differential equations

$$P_1 u(t) = P_2 u(t) = 0$$

with

$$\begin{aligned} P_1 &= t^3\partial_t^5 + (2t^4 + 17t^2)\partial_t^4 + (32t^3 + 80t)\partial_t^3 \\ &\quad + (-4t^4 + 144t^2 + 100)\partial_t^2 + (-28t^3 + 192t)\partial_t - 36t^2 + 48, \\ P_2 &= t^3\partial_t^4 + (3t^4 + 14t^2)\partial_t^3 + (2t^5 + 35t^3 + 52t)\partial_t^2 \\ &\quad + (14t^4 + 102t^2 + 48)\partial_t + 18t^3 + 66t. \end{aligned}$$

6 (Appendix) Holonomic systems

Let us present a precise definition of holonomicity. We define the total or the $(\mathbf{1}, \mathbf{1})$ -order of nonzero $P \in D_n$ to be

$$\text{ord}_{(\mathbf{1}, \mathbf{1})}(P) := \max\{|\alpha| + |\beta| = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n \mid a_{\alpha, \beta} \neq 0\}.$$

We set $\text{ord}_w(0) := -\infty$. This induces the filtration

$$F_k(D_n) := \{P \in D_n \mid \text{ord}_{(\mathbf{1}, \mathbf{1})}(P) \leq k\} \quad (k \in \mathbb{Z})$$

on the ring D_n . Let M be a left D_n -module and $\{F_k(M)\}_{k \in \mathbb{Z}}$ be a good $(\mathbf{1}, \mathbf{1})$ -filtration. This means the following properties:

- (1) every $F_k(M)$ is a finite dimensional vector space over \mathbb{C} ;
- (2) $F_k(M) \subset F_{k+1}(M)$ for all $k \in \mathbb{Z}$;
- (3) $\bigcup_{k \in \mathbb{Z}} F_k(M) = M$;
- (4) $F_i(D_n)F_k(M) \subset F_{i+k}(M)$ for all $i, k \in \mathbb{Z}$;
- (5) there exists $k_1 \in \mathbb{Z}$ such that $F_k(M) = 0$ for $k \leq k_1$;
- (6) there exists $k_2 \in \mathbb{Z}$ such that $F_i(D_n)F_k(M) = F_{i+k}(M)$ for $k \geq k_2$.

Then there exists a polynomial in k such that $\dim_{\mathbb{C}} F_k(M) = p(k)$ for sufficiently large k . The degree of $p(k)$ does not depend on the choice of a good $(\mathbf{1}, \mathbf{1})$ -filtration of M and is called the dimension of the module M , which we denote by $d(M)$. It was proved by Bernstein [1] that $d(M) \geq n$ if $M \neq 0$. The following definition is due to Bernstein [1]:

Definition 1 A finitely generated left D_n -module M is called a *holonomic system* if $d(M) \leq n$. We also call a left ideal I of D_n to be a *holonomic ideal*, by abuse of terminology, if the left D_n -module D_n/I is holonomic.

Note that $d(M) \leq n$ is equivalent to $d(M) = n$ or $M = 0$ in view of the Bernstein inequality stated above. The dimension $d(M)$ can be computed as the degree of the Hilbert function from a Gröbner base with respect to a term order which is compatible with the total degree.

Holonomicity is preserved by operations such as sum, product, restriction to affine subvarieties, and integration with respect to some of the variables (cf. [1], [2]) and they are computable (see e.g., [10]). Let $R_n := \mathbb{C}(x)\langle \partial_1, \dots, \partial_n \rangle$ be the ring of differential operators with rational function coefficients. A D_n -module M is said to be of finite rank and the dimension is called the rank of M , if $R_n M$ is a finite dimensional vector space over $\mathbb{C}(x)$. A holonomic D_n -module M is of finite rank but the converse is not true in general. Note that there is an algorithm for a given D_n -module M of finite rank to construct a holonomic D_n -module \tilde{M} and a surjective D_n -homomorphism of M to \tilde{M} ([12],[14]). If M is a system of differential equations of finite rank for an analytic function u , then we have an isomorphism $\tilde{D}_n / \text{Ann}_{D_n} u$.

Example 3 Set $f = x^2y^2 + z^2$ and consider the function f^{-1} . It is easy to see that the operators

$$\begin{aligned} f\partial_x + \frac{\partial f}{\partial x} &= (x^2y^2 + z^2)\partial_x + 2xy^2, \\ f\partial_y + \frac{\partial f}{\partial y} &= (x^2y^2 + z^2)\partial_y + 2x^2y, \\ f\partial_z + \frac{\partial f}{\partial z} &= (x^2y^2 + z^2)\partial_z + 2z \end{aligned}$$

annihilate f^{-1} . Let J be the left ideal generated by these three operators, a ‘naive’ annihilator. Then the Hilbert function of the D_3/J is

$$\frac{1}{30}k^5 + \frac{1}{4}k^4 + \frac{7}{6}k^3 + \frac{5}{4}k^2 + \frac{43}{10}k$$

which means that the degree of D_3/J is 5 and hence D_3/J is not holonomic although it is of rank one. The true annihilator I of f^{-1} is generated by

$$\begin{aligned} 3z^2\partial_x^2 - 2y^3\partial_z\partial_y - 2y^2\partial_z, \\ 3z^2\partial_y - 2yx^2\partial_z, \\ 3z^2\partial_x - 2y^2x\partial_z, \\ 3y\partial_y + 2z\partial_z + 6, \\ -x\partial_x + y\partial_y \end{aligned}$$

and the Hilbert function of D_3/I is

$$\frac{7}{3}k^3 - \frac{3}{2}k^2 + \frac{43}{6}k - 1,$$

which implies that D_3/I is holonomic. The Hilbert function of $D_3/\text{Ann}_{D_3} \log f$ is

$$2k^3 + \frac{3}{2}k^2 + \frac{5}{2}k - 1.$$

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